MA3205 Set Theory

- cardinal → size
- ordinal → order
- $\mathbb{N} := \{0, 1, 2, \dots\}$

- Zermelo-Fraenkel set theory: not everything is a set, only those expressible as $\{x \in z : P(x)\}$ where z is a set is a set.
  - Some axioms:
    - Set existence: $\exists x (x = x)$
    - Extensionality: $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$
    - Comprehension scheme: For any formula $\varphi(x)$: $\forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \wedge \varphi(x)))$
    - Pairing: $\forall x \forall y \exists z (x \in z \wedge y \in z)$
    - Union
    - Replacement
    - Infinity
    - Foundation
    - Choice
- Unordered pair: $\{a, b\}$
- Ordered pair: $(a,b) := \{\{a\}, \{a,b\}\}$
  - Thm: $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$
- Cartesian product: $A \times B := \{(a,b) : a \in A \wedge b \in B\}$
- Binary relation: a set of ordered pairs
  - $\mathrm{dom}(R) := \{x : \exists y\, (x,y) \in R\}$
  - $\mathrm{ran}(R) := \{y : \exists x (x,y) \in R\}$
- Function: a binary relation s.t. $\forall x$ there is at most one y s.t. $(x,y) \in f$. (i.e. $(x,y) \in f \wedge (x,z) \in f \rightarrow y = z$)
  - $f : A \rightarrow B$: $\mathrm{dom}(f) = A$, $\mathrm{ran}(f) \subseteq B$
  - injective: $f(x) = y$ and $f(x') = y \Rightarrow x = x'$
  - surjective: $\mathrm{ran}(f) = B$
  - bijective: injective & surjective
  - B^A : set of all functions from A to B.
  - composition: $g \circ f$: $(g \circ f)(x) = g(f(x))\ \forall x \in A$.
  - $f[X'] := \{y : (\exists x \in X')\, f(x) = y\}$ where $X' \subseteq X$ (the image of X' under f)
  - $f^{-1}[Y'] := \{x : f(x) \in Y'\}$ where $Y' \subseteq Y$ (the pre-image of Y' under f)
  - $f^{-1} := \{(y,x) : (x,y) \in f\}$. f^{-1} is a relation, but not neccessarily a function
- Characteristic function: Given a fixed nonempty set X, and $A \subseteq X$: $\chi_A : X \rightarrow \{0,1\}$
  $x \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$
- $S := \langle S_i : i \in I \rangle$ is a function with domain I (it maps i to S_i) called an indexed system of sets.
  - A is indexed by S : $A = \{S_i : i \in I\} = \mathrm{ran}(S)$
  - $\prod S := \{f : \mathrm{dom}(f) = I \wedge (\forall i \in I)\ f(i) \in S\}$ is the product of S
- Given $R \subseteq A \times A$ (i.e. R is a binary relation in A):
  - R is reflexive: $\forall a \in A : aRa$
  - R is symmetric: $\forall a, b \in A : aRb \Rightarrow bRa$
  - R is transitive: $\forall a,b,c \in A : aRb$ and $bRc \Rightarrow aRc$
  - R is an equivalence: R is reflexive, symmetric, & transitive

  note: these properties depend on the chosen A.

- Equivalence class : $[a]_E := \{x \in A : xEa\}$ where E is an equivalence on A and $a \in A$
  - $A/E = \{[a]_E : a \in A\}$ (set of all equivalence classes)
  - A/E is a partition of A
- Partition : S is a partition of A if S is a system of nonempty mutually disjoint sets whose union is A.
- Given a partition S, we can construct an equivalence relation $E_S = \{(a,b) \in A \times A : (\exists C \in S)\ a \in C \wedge b \in C\}$
- $X \subseteq A$ is a set of representatives for S if $\forall C \in S$, $X \cap C = \{a\}$ for some a
  (i.e. exactly one element per set in the partition)
- Equivalence $\cong$ Partition : Given a set A:
  - If E is an equivalence, then $E_{A/E} = E$
  - If S is a partition, $A/E_S = S$
- Antisymmetric : binary relation s.t. $\forall a,b \in A$, aRb and $bRa \Rightarrow a=b$
- Partial ordering : binary relation that is reflexive, antisymmetric, transitive.
- Asymmetric : $aSb \Rightarrow \neg bSa$
- Strict partial ordering : asymmetric + transitive
- Linear/total ordering : $\forall x \forall y [x \leq y \vee y \leq x]$ (i.e. any two elements are comparable)
- Definitions for orderings : Given a partial ordering of A, and $B \subseteq A$:
  - $b \in B$ is the least element of B : $b \leq x\ \forall x \in B$
  - $b \in B$ is a minimal element of B : there is no $x \in B$ s.t. $x \leq b$ and $x \neq b$
  - $a \in A$ is a lower bound of B : $a \leq x\ \forall x \in B$
  - $a \in A$ is the infimum of B : the greatest element in the set of all lower bounds of B
- $A \subseteq S$ is closed under f : $\forall x,y \in A$ s.t. $f(x,y)$ is defined, $f(x,y) \in A$
- Isomorphism : a relabelling of the set s.t. all relations & functions are preserved
- Antichain of a partially ordered set : a subset where no two elements are comparable
- Chain of a partially ordered set : a subset where any two elements are comparable
- CAC thm : every infinite partially ordered set has an infinite chain or infinite antichain.
- Integers : $n := \{0, 1, \dots, n-1\}$
  - Successor : $S(x) := x \cup \{x\}$
- Inductive set : $\begin{cases} \cdot\ \phi \in I \\ \cdot \text{ Closed under } S \text{ (i.e. } \forall a \in I,\ S(a) \in I) \end{cases}$
- Axiom of infinity : an inductive set exists
- Natural numbers : $\mathbb{N} := \{x : x \in I \text{ for every inductive set } I\}$
  - Lemma : $\mathbb{N}$ is an inductive set.
- Induction principle : Let $P(x)$ be a property.
  $$\left.\begin{array}{l} \cdot\ P(0) \text{ holds} \\ \cdot\ \forall n \in \mathbb{N},\ P(n) \Rightarrow P(S(n)) \end{array}\right\} \Rightarrow \forall n \in \mathbb{N},\ P(n)$$
- Ordering on $\mathbb{N}$: $m < n := m \in n$
  - Lemmas : $\cdot\ \forall n \in \mathbb{N},\ 0 \leq n$
    - $\forall k,n \in \mathbb{N}$, $k < \underbrace{n+1}_{S(n)} \Leftrightarrow k < n$ or $k = n$

  Thm : $(\mathbb{N}, <)$ is a linearly ordered set
- Strong induction : Let $P(x)$ be a property.
  - $\forall n \in \mathbb{N}$, $((\forall k < n, P(k)) \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)$
- Well-ordering : every non-empty subset has a least element
  - $(\mathbb{N}, <)$ is a well-ordered set

- Recursion thm on $\mathbb{N}$: Given a set A, an element $a \in A$, a function $g: A \times \mathbb{N} \to A$, there exists a unique function $f: \mathbb{N} \to A$ s.t. $\begin{cases} \cdot f(0) = a \\ \cdot \forall n \in \mathbb{N},\ f(n+1) = g(f(n), n) \end{cases}$

  Parametric version: $a: P \to A$; $g: P \times A \times \mathbb{N} \to A$; $f: P \times \mathbb{N} \to A$
  - $f(p, 0) = a(p) \quad \forall p \in P$
  - $f(p, n+1) = g(p, f(p,n), n), \ \forall n \in \mathbb{N}\ \forall p \in P$
- Union of sets: $\bigcup A := \{ b : \exists a (a \in A \wedge b \in a)\}$
  - axiom of union: $\bigcup A$ is a set

  Intersection: $\bigcap A = \{ b : \forall a (a \in A \to b \in a)\}$
- Addition on $\mathbb{N}$: There is a unique operation $+: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ s.t. $\begin{cases} \cdot +(0, m) = m \ \forall m \in \mathbb{N} \\ \cdot +(m, S(n)) = S(+(m,n)) \ \forall m, n \in \mathbb{N} \end{cases}$
  - $+$ is commutative (i.e. $\forall n, m \in \mathbb{N},\ m+n = n+m$)
- Closure of a set: Given structure $\mathcal{A} = (A, f, g)$ where $f: A \times A \to A$ and $g: A \to A$
  - $\overline{C} := \bigcap \{ B \subseteq A : C \subseteq B$ and B is closed under f and $g \}$
  - $\underline{C} := \bigcup \{ C_n : n \in \mathbb{N} \}$ where $\begin{cases} C_0 = C \\ C_{n+1} = C_n \cup f[C_n \times C_n] \cup g[C_n] \end{cases}$
  - Thm: $\overline{C} = \underline{C}$
- Cardinality
  - $|A| \le |B|$:= there is a one-to-one mapping from A to B
  - $|A| = |B|$ (equipotent) : there is a one-to-one & onto function from A to B
  - $|A| < |B| \Leftrightarrow |A| \le |B|$ and $|B| \not\le |A|$
  - $|A| < |B| \Leftrightarrow |A| \le |B|$ and $|A| \ne |B|$
- Cantor-Bernstein thm: $|X| \le |Y|$ and $|Y| \le |X| \implies |X| = |Y|$
  - Lemma: $A_1 \subseteq B \subseteq A$ and $|A_1| = |A| \implies |B| = |A|$
- Finite set: $|S| = n$ for some $n \in \mathbb{N}$.
- Infinite set: not finite
- Lemmas of finite sets:
  - If $n \ne m$, there is no bijection between n and m.
  - If $|S| = n$ and $|S| = m$ then $n = m$
  - $\mathbb{N}$ is infinite
- Countable: $|S| = |\mathbb{N}|$
- At most countable: $|S| \le |\mathbb{N}|$
- Thm: any subset of a countable set is either finite or countable
- Countable union of countable sets is countable (requires AC)
- $Seq(A)$: set of finite sequences of elements of A.
- A is countable $\Rightarrow Seq(A)$ is countable
- $\aleph_0 := |\mathbb{N}|$
- Cantor's thm: $|2^{\mathbb{N}}| = |P(\mathbb{N})| > |\mathbb{N}|$
- $(A, <)$ and $(B, \prec)$ are similar (i.e. have the same order type) if they are isomorphic (note: we are talking about linear orders)

  an equivalence relation
- Finite linearly-ordered sets with the same size are similar
- $(X, <)$ is dense : X has at least 2 elements and $\forall a, b \in X,\ a < b \Rightarrow \exists x \in X$ s.t. $a < x < b$
- Thm dense linear orderings: Any two countable dense linearly ordered sets without endpoints are similar.
- Thm even for non-dense countable linear orderings: Every countable linearly ordered set is isomorphic to any countable dense linearly ordered set without endpoints.
- Integers: $\mathbb{Z}$:= set of all equivalence classes of $\mathbb{N} \times \mathbb{N}$ modulo $\sim$

  $(a,b) \sim (c,d) := a+d = b+c$
  - $[(a,b)] <_{\mathbb{Z}} [(c,d)] := a+d <_{\mathbb{N}} b+c$
  - $\mathbb{Z}$ is countable
  - $[(a,b)] +_{\mathbb{Z}} [(c,d)] = [(a+c, b+d)]$
  - $[(a,b)] \cdot_{\mathbb{Z}} [(c,d)] = [(ac+bd, ad+bc)]$
  - $-[(a,b)] = [(b,a)]$
- Embedding of A into B: an injective function $f: A \to B$ that preserves properties
  - E.g. embedding $\mathbb{N}$ into $\mathbb{Z}$: $f(n) := [(n, 0)]$ is injective & order-preserving
- Rational numbers: $\mathbb{Q}$:= set of equivalence classes of $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ modulo $\sim$

  $(a,b) \sim (c,d) := ad = bc$
  - $\mathbb{Q}$ is countable
  - $(\mathbb{Q}, <_{\mathbb{Q}})$ is a dense linearly ordered set without endpoints
- Complete dense linearly ordered set $(P, <)$: every $S \subseteq P$ bounded from above has a supremum in P.
  - $\mathbb{Q}$ is not complete (e.g. because $\{x \in \mathbb{Q}, x^2 < 2\}$ has no supremum)
- Dedekind cut: subset $A \subseteq \mathbb{Q}$ s.t. :
  (1) $A \ne \emptyset$ and $A \ne \mathbb{Q}$
  (2) $p \in A$ and $q < p \implies q \in A$
  (3) A does not have a greatest element

- **Real numbers**: $\mathbb{R} :=$ set of all Dedekind cuts in $\mathbb{Q}$.
  - $A <_{\mathbb{R}} B := A \subset B$
  - $i: \mathbb{Q} \to \mathbb{R} : p \mapsto \{q \in \mathbb{Q} : q < p\}$ is injective and order-preserving
  - $\mathbb{Q}$ is dense in $\mathbb{R}$
  - $\mathbb{R}$ does not have endpoints
  - $\mathbb{R}$ is complete
  - $A +_{\mathbb{R}} B := \{p +_{\mathbb{Q}} q : p \in A \text{ and } q \in B\}$
  - $A \cdot_{\mathbb{R}} B := \begin{cases} 0_{\mathbb{R}} \cup \{rs : 0 \le r \in |x| \text{ and } 0 \le s \in |y|\} & \text{if } x \text{ and } y \text{ are both nonnegative or both negative} \\ -(|A| \cdot_{\mathbb{R}} |B|) & \text{otherwise} \end{cases}$
- **Open & Closed subsets of $\mathbb{R}$**:
  - $A \subseteq \mathbb{R}$ is open : $\forall a \in A, \exists \delta > 0$ s.t. $\underbrace{\forall x \in \mathbb{R}, (|x - a| < \delta \Rightarrow x \in A)}_{\text{i.e. } (a-\delta, a+\delta) \subseteq A}$
  - $B \subseteq \mathbb{R}$ is closed : $\mathbb{R} \setminus B$ is open
- Every system of mutually disjoint open intervals in $\mathbb{R}$ is at most countable
- Every open set is a union of a system of open intervals with rational endpoints
- $a \in \mathbb{R}$ is an accumulation point of $A \subseteq \mathbb{R}$: $\forall \delta > 0, \exists x \in A \setminus \{a\}$ s.t. $|x - a| < \delta$
- $a \in A$ is an isolated point of $A \subseteq \mathbb{R}$: $\exists \delta > 0$ s.t. $\forall x \in A \setminus \{a\}, |x - a| \ge \delta$
- $A \subseteq \mathbb{R}$ is closed $\Leftrightarrow$ all accumulation points of A belong to A.
- $A \subseteq \mathbb{R}$ is perfect: $A \neq \emptyset$ and A is closed without isolated points.
- **Cantor set**: $F := \bigcap_{n \in \mathbb{N}} F_n$ where $F_n := \bigcup \{D_s : s \in \{0,1\}^n\}$ where $\begin{cases} D_{\langle\rangle} = [0,1] \\ D_{\langle s_0, \ldots, s_{n-1}, 0\rangle} = [a, a + \frac{1}{3}(b-a)] \\ D_{\langle s_0, \ldots, s_{n-1}, 1\rangle} = [a + \frac{2}{3}(b-a), b] \end{cases}$
  - $|F| = 2^{\aleph_0}$ (Pf: Find a bijection from $\{0,1\}^{\mathbb{N}}$ onto F)
  - F is perfect
  - $[0,1] \setminus F$ is dense in $[0,1]$
- **Cardinal addition**: If $\kappa = |A|$ and $\lambda = |B|$ and $A \cap B = \emptyset$ then $\kappa + \lambda = |A \cup B|$
- **Cardinal multiplication**: If $\kappa = |A|$ and $\lambda = |B|$ then $\kappa \cdot \lambda = |A \times B|$
- **Cardinal exponentiation**: If $\kappa = |A|$ and $\lambda = |B|$ then $\kappa^\lambda = |A^B|$
- **Lemmas**:
  - $\kappa + \lambda = \lambda + \kappa$, $\kappa \cdot \lambda = \lambda \cdot \kappa$ } Commutativity of addition & multiplication
  - $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$, $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$ } Associativity of addition & multiplication
  - $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$ } Distributivity of addition over multiplication
  - $\kappa^{\lambda + \mu} = \kappa^\lambda \cdot \kappa^\mu$
  - $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$
  - $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$
- **Cantor's thm**: For any set X: $|X| \stackrel{..}{<} |P(X)| = |2^X|$
- $|\mathbb{R}| = 2^{\aleph_0}$
- **Theorems**:
  - $n + 2^{\aleph_0} = \aleph_0 + 2^{\aleph_0} = 2^{\aleph_0} + 2^{\aleph_0} = 2^{\aleph_0}$ $\quad \forall n \in \mathbb{N}$
  - $n \cdot 2^{\aleph_0} = \aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$ $\quad \forall n \in \mathbb{N} \setminus \{0\}$
  - $(2^{\aleph_0})^n = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ $\quad \forall n \in \mathbb{N} \setminus \{0\}$
  - $n^{\aleph_0} = \aleph_0^{\aleph_0} = 2^{\aleph_0}$ $\quad \forall n \in \mathbb{N} \setminus \{0,1\}$
- **Thm**: $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$

- The set of all open subsets of $\mathbb{R}$ has cardinality $2^{\aleph_0}$.
- The set of all functions $\mathbb{R}\to\mathbb{R}$ has cardinality $2^{2^{\aleph_0}} > 2^{\aleph_0}$
- Continuum Hypothesis: There is no uncountable cardinal κ s.t. $\kappa < 2^{\aleph_0}$
- Every perfect set has cardinality $2^{\aleph_0}$
- Lemmas about perfect sets:
  - Every perfect set contains two smaller perfect sets (For each perfect set F, $\exists A,B\subseteq F$ s.t. $A\cap B=\phi$ and A,B are perfect)
  - Every perfect set contains a perfect subset of arbitrarily small diameter (For each perfect set F and each $n\in\mathbb{N}$, $\exists H=H(F,n)\subseteq F$ that is perfect and has diameter $\le \frac{1}{n}$.

    $\mathrm{diam}(A) := \sup(A) - \inf(A)$
- Every uncountable closed set contains a perfect subset
- Every closed set has at most countably many isolated points
- Derivative of a set $A\subseteq\mathbb{R}$: $A' :=$ set of all accumulation points of A.
  - A is closed $\iff A'\subseteq A$
  - A is perfect $\iff A\neq\phi$ and $A'=A$.
  - A' is closed
  - If F is closed, then $F-F'$ is at most countable
- Cantor derivatives: $A^{(0)} = A$, $A^{(\alpha+1)} = (A^{(\alpha)})'$, $A^{(\alpha)} = \bigcap_{\xi<\alpha} A^{(\xi)}$ where α is a limit
- Initial segment S of a linearly ordered set $(L,<)$: $S\subsetneq L$ and $\forall a\in S, (x<a \Rightarrow x\in S)$
- $(W,<)$ is well-ordered: every non-empty subset of W has a least element.
  - $(\mathbb{N},<)$ is well-ordered
  - $(\mathbb{Z},<)$ is not well-ordered
- Initial segment being determined by a single element: If $(W,<)$ is a well-ordered set and S is an initial segment of $(W,<)$, then $\exists a\in W$ s.t. $S=\{x\in W : x<a\}$.
- Increasing function $f: W\to W$ on well-ordered set W: For all $x\in W$, $f(x)\geq x$

  Pf: By showing that $X:=\{x \mid f(x)<x\}$ (if not, then let a be the least element in X, so $f(a)<a$, so $f(f(a))<f(a)$. Contradiction.)

  Cor:
  - no well-ordered set is isomorphic to an initial segment of itself
  - each well-ordered set has only one automorphism (the identity function)
  - if $\underbrace{(W_1,<)\cong(W_2,<)}_{\text{wellordered}}$ then the isomorphism is unique
- Isomorphism thm of well-ordered sets: If $(W_1,<)$ and $(W_2,<)$ are well-ordered sets then exactly one of the following holds:
  - W_1 and W_2 are isomorphic
  - W_2 is isomorphic to an initial segment of W_1
  - W_1 is isomorphic to an initial segment of W_2

  in any case, the isomorphism is unique

Set T is transitive: $u\in v\in T \Rightarrow u\in T$ (i.e. every element of T is also a subset of T)

Set α is an ordinal: α is transitive & α is well ordered by $\in_\alpha$
- every natural number is an ordinal
- $\omega=\mathbb{N}$ is an ordinal
- $\alpha+1 := S(\alpha) := \alpha\cup\{\alpha\}$ is an ordinal
- an ordinal α is a successor ordinal if $\exists\beta$ s.t. $\alpha=\beta+1$
- an ordinal α is a limit ordinal otherwise
- $\alpha<\beta := \alpha\in\beta$
  - satisfies asymmetry & transitivity, so it is a strict partial order
  - satisfies trichotomy (exactly one of $\alpha<\beta$, $\alpha=\beta$, $\beta<\alpha$ holds), so it is a linear order
  - every nonempty subset has a least element, so it is a well-order

- To show that α is a limit ordinal, it suffices to show that $\forall\beta<\alpha$, $\beta+1<\alpha$.

Lemmas:
- For any ordinal α, $\alpha\notin\alpha$
- For any ordinal α, $\forall x\in\alpha$, x is an ordinal
- For any ordinals α & β, $\alpha\subset\beta \Rightarrow \alpha\in\beta$

- Ordinals is a class: For any set of ordinals X, $\exists$ ordinal α s.t. $\alpha\notin X$ (i.e. "the set of all ordinals" does not exist)

- W_1 has smaller order type than W_2 : $W_1 \cong W_2[a]$ for some $a \in W_2$
  well-ordered sets
- Axiom schema of replacement: If for every x there is a unique y in which $P(x,y)$ holds :
  then for every set A, there is a set B s.t. $\forall x \in A, \exists y \in B$ s.t. $P(x,y)$ holds.
- Every well-ordered set is isomorphic to a unique ordinal
- Transfinite induction principle : Given a property $P(x)$:
  If for any ordinal α, $\left((\forall \beta < \alpha : P(\beta)) \Rightarrow P(\alpha) \right)$
  then $P(\alpha)$ holds for all ordinals α.
  kind of like strong induction
- Transfinite induction principle (easier version): Given a property $P(x)$:
  If: · $P(0)$ holds
  · $P(\alpha) \Rightarrow P(\alpha+1)$ for all ordinal α
  · For all limit ordinals α, if $P(\beta)$ holds for all $\beta < \alpha$, then $P(\alpha)$ holds
  Then $P(\alpha)$ holds for all ordinals α.
- Transfinite recursion theorem: If G is an operation defined on the class of all sets, then there is a unique operation F defined on the class of all ordinals s.t. for all ordinals α, $F(\alpha) = G(F \restriction \alpha)$
  (in other words, G is used to "generate" each $F(\alpha)$ from the $F(\beta)$ where $\beta < \alpha$
- Transfinite recursion theorem (easier version): If G_1, G_2, G_3 are operations, then there is a unique operation F defined on the class of all ordinals s.t. :
  · $F(0) = G_1()$
  · $F(\alpha+1) = G_2(\alpha, F(\alpha))$
  · $F(\alpha) = G_3(\{(\beta, F(\beta)) : \beta < \alpha\})$ if $\alpha \neq 0$ is a limit ordinal
- Operations on ordinals:
  - Addition :
    - $\beta + 0 = \beta$
    - $\beta + (\alpha+1) = (\beta+\alpha)+1$
    - $\beta + \alpha = \sup\{\beta+\gamma : \gamma < \alpha\}$ for all limit $\alpha \neq 0$ (note : $1+\omega = \omega \neq \omega+1$)
    - $\alpha_1 < \alpha_2 \iff \beta+\alpha_1 < \beta+a_2$ (left cancellation law)
    - $\alpha_1 = \alpha_2 \iff \beta+\alpha_1 = \beta+\alpha_2$
    - $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ (associative law)
  - Multiplication:
    - $\beta \cdot 0 = 0$
    - $\beta \cdot (\alpha+1) = \beta\cdot\alpha + \beta$
    - $\beta \cdot \alpha = \sup\{\beta\cdot\gamma : \gamma < \alpha\}$ for all limit $\alpha \neq 0$ (note : $2\cdot\omega = \omega \neq \omega\cdot 2 = \omega\cdot 1 + \omega = \omega+\omega$)
  - Exponentiation:
    - $\beta^0 = 1$
    - $\beta^{\alpha+1} = \beta^\alpha \cdot \beta$
    - $\beta^\alpha = \sup\{\beta^\gamma : \gamma < \alpha\}$ for all limit $\alpha \neq 0$ (note : $2^\omega = \sup\{2^n : n < \omega\} = \omega$)
      cardinal exponentiation is totally different!
- α is an initial ordinal : α is not equipotent to any $\beta < \alpha$.
  - E.g. all the natural numbers, ω, etc
  - but not $\omega+1$, $\omega+\omega$, etc
- Cardinal number of a well-orderable set X : the unique initial ordinal in which X is equipotent to.
- Hartogs number of A : $h(A)$:= least ordinal which is not equipotent to any (not necessarily proper) subset of A.
  (i.e. least ordinal for which there is no injection to A)
  - $h(A)$ is always an initial ordinal
  - $h(A)$ exists for all A.
  - $\omega_0 := \omega$
  - $\omega_{\alpha+1} := h(\omega_\alpha)$
  - $\omega_\alpha := \sup\{\omega_\beta : \beta < \alpha\}$ if α is a limit ordinal

- <u>From ordinals to cardinals</u>:
  - for all ordinal α, ω_α is an infinite initial ordinal
  - for all ordinal α, $\alpha \le \omega_\alpha$
  - if Ω is an infinite initial ordinal, then $\Omega = \omega_\alpha$ for some α.
- $\aleph_\alpha = \omega_\alpha$
- $(\alpha_1, \alpha_2) \prec (\beta_1, \beta_2) := \begin{cases} \max\{\alpha_1,\alpha_2\} < \max\{\beta_1,\beta_2\} & \text{or} \\ \max\{\alpha_1,\alpha_2\} = \max\{\beta_1,\beta_2\} \text{ and } \alpha_1 < \beta_1 & \text{or} \\ \max\{\alpha_1,\alpha_2\} = \max\{\beta_1,\beta_2\} \text{ and } \alpha_1 = \beta_1 \text{ and } \alpha_2 < \beta_2 \end{cases}$

  is a well-ordering

- $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ for every α.
  - $\aleph_\alpha \cdot \aleph_\beta = \aleph_\beta$ for every $\alpha \le \beta$
  - $n \cdot \aleph_\alpha = \aleph_\alpha$ for every α
  - $\aleph_\alpha + \aleph_\beta = \aleph_\beta$ for every $\alpha \le \beta$
  - $n + \aleph_\alpha = \aleph_\alpha$ for every α
- <u>Choice function</u> on S: any function g s.t. $g(X) \in X$ for all nonempty $X \in S$.
  - i.e. it chooses one element from the set.
- <u>Thm well-ordering given choice</u>: If $P(A)$ has a choice function, then A can be well-ordered.

  <u>Pf</u>: $G(x) := \begin{cases} g(A - \mathrm{ran}(x)), & \text{if } A - \mathrm{ran}(x) \neq 0 \\ a, & \text{otherwise} \end{cases}$
- <u>Thm choice given well-ordering</u>: If A can be well-ordered, then $P(A)$ has a choice function.

  <u>Pf</u>: $g(x) := \begin{cases} \text{the } \prec\text{-least element of } x, & \text{if } x \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$
- Every finite system of sets has a choice function
  - <u>Pf</u>: By induction on the number of sets in the system
- <u>Axiom of Choice</u>: there is a choice function for every system of sets
- <u>TFAE</u>:
  - Axiom of choice
  - Every partition has a set of representatives
  - Every set can be well-ordered
  - Zorn's lemma
- <u>Zorn's Lemma</u> If every chain in a partially ordered set has an upper bound, then the partially ordered set has a maximal element
  - In a partially ordered set $(X, <)$, a subset $C \subseteq X$ is a <u>chain</u> in X iff $(C, <)$ is linearly ordered
- <u>Consequences of Axiom of Choice</u>:
  - Every infinite set A has a countable subset C
  - For every infinite set S there exist a unique $\aleph_\alpha$ s.t. $|S| = \aleph_\alpha$
  - For any sets A and B, either $|A| \le |B|$ or $|B| \le |A|$
  - For any function f and set A, $|f[A]| \le |A|$

    f restricted to A
  - Every vector space has a basis
- <u>Decomposition of closed set into perfect set</u>: If F is a closed set of reals, then there exists an at most countable ordinal Θ s.t.
  - (a) For every $\alpha < \Theta$, $F^{(\alpha)} \setminus F^{(\alpha+1)}$ is nonempty & at most countable
  - (b) $F^{(\Theta+1)} = F^{(\Theta)}$
  - (c) $F^{(\Theta)}$ is either empty or perfect
  - (d) $F \setminus F^{(\Theta)}$ is at most countable
- Every uncountable closed set can be decomposed into a perfect set and an at most countable set.